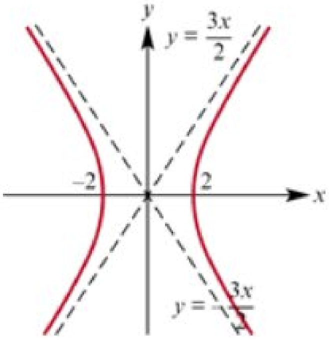
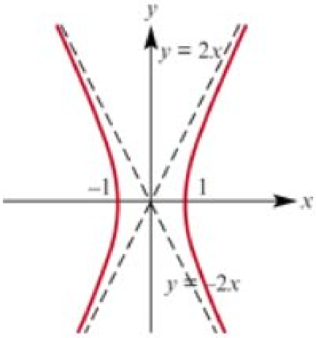
Exercise 12F: Solutions

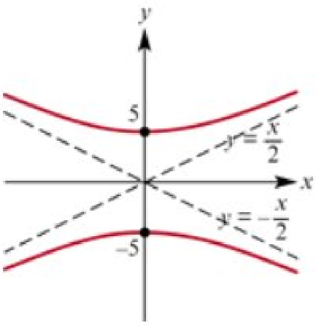
1 a



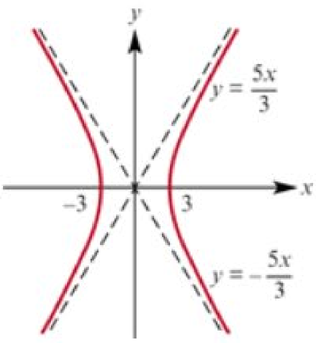
b



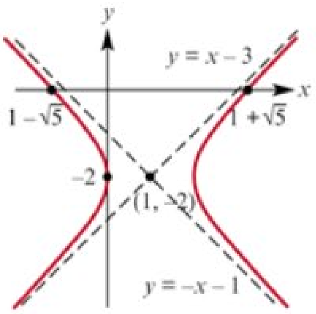
c



d



2 a



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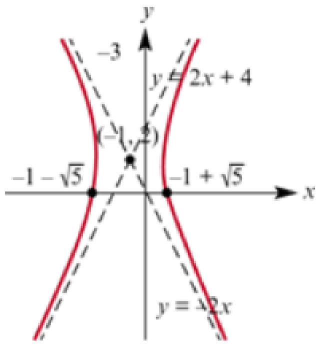

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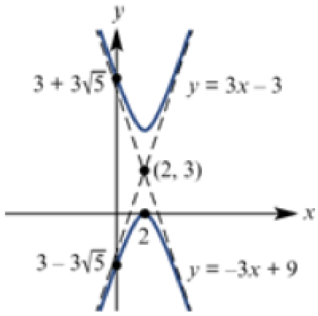
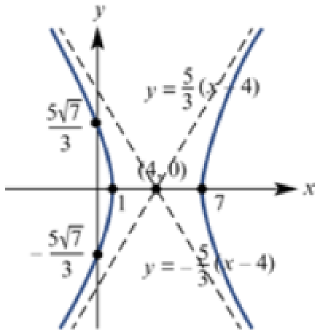
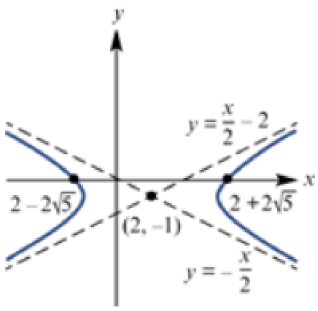


d



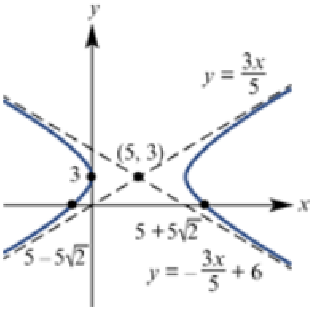
e For this question, we must first complete the square in both x and y variables. This gives,

$$\begin{aligned} x^2 - 4y^2 - 4x - 8y - 16 &= 0 \\ (x^2 - 4x) - 4(y^2 + 2y) - 16 &= 0 \\ (x^2 - 4x + 4 - 4) - 4(y^2 + 2y + 1 - 1) - 16 &= 0 \\ ((x-2)^2 - 4) - 4((y+1)^2 - 1) - 16 &= 0 \\ (x-2)^2 - 4 - 4(y+1)^2 + 4 - 16 &= 0 \\ (x-2)^2 - 4(y+1)^2 &= 16 \\ \frac{(x-2)^2}{16} - \frac{(y+1)^2}{4} &= 1 \end{aligned}$$



f For this question, we must first complete the square in both x and y variables. This gives,

$$
\begin{aligned}
9x^2 - 25y^2 - 90x + 150y &= 225 \\
9(x^2 - 10x) - 25(y^2 - 6y) &= 225 \\
9(x^2 - 10x + 25 - 25) - 25(y^2 - 6y + 9 - 9) &= 225 \\
9((x-5)^2 - 25) - 25((y-3)^2 - 9) &= 225 \\
9(x-5)^2 - 225 - 25(y-3)^2 + 225 &= 225 \\
9(x-5)^2 - 25(y-3)^2 &= 225 \\
\frac{(x-5)^2}{25} - \frac{(y-3)^2}{9} &= 1
\end{aligned}
$$



3 Let (x, y) be the coordinates of point P. If $AP - BP = 6$, then

$$\sqrt{(x-4)^2 + y^2} - \sqrt{(x+4)^2 + y^2} = 3$$

$$\sqrt{(x-4)^2 + y^2} = 6 + \sqrt{(x+4)^2 + y^2}.$$

Squaring both sides gives

$$(x-4)^2 + y^2 = 36 + 12\sqrt{(x+4)^2 + y^2} + (x+4)^2 + y^2$$

Expanding and simplifying

$$
\begin{aligned}
x^2 - 8x + 16 + y^2 &= 36 + 12\sqrt{(x+4)^2 + y^2} + x^2 + 8x + 16 + y^2 \\
-16x - 36 &= 12\sqrt{(x+2)^2 + y^2} \\
-4x - 9 &= 3\sqrt{(x+4)^2 + y^2}
\end{aligned}
$$

Note that this only holds if $x \le -\frac{9}{4}$. Squaring both sides agains gives,

$$16x^2 + 72x + 81 = 9(x^2 + 8x + 16 + y^2)$$

Expanding and simplifying yields

$$
\begin{aligned}
16x^2 + 72x + 81 &= 9x^2 + 72x + 144 + 9y^2 \\
7x^2 - 9y^2 &= 63 \\
\frac{x^2}{9} - \frac{y^2}{7} &= 1, \quad x \le -\frac{9}{4}.
\end{aligned}
$$

4 Let (x, y) be the coordinates of point P. If $AP - BP = 4$, then

$$\sqrt{(x+3)^2+y^2} - \sqrt{(x-3)^2+y^2} = 4$$

$$\sqrt{(x+3)^2+y^2} = 4 + \sqrt{(x-3)^2+y^2}$$

Squaring both sides gives

$$(x+3)^2+y^2 = 16 + 8\sqrt{(x-3)^2+y^2} + (x-3)^2+y^2$$

Expanding and simplifying

$$x^2+6x+9+y^2 = 16+8\sqrt{(x-3)^2+y^2}+x^2-6x+9+y^2$$

$$12x-16 = 8\sqrt{(x-3)^2+y^2}$$

$$3x-4 = 2\sqrt{(x-3)^2+y^2}$$

Note that this only holds if $x \geq \frac{4}{3}$. Squaring both sides again gives,

$$9x^2-24x+16 = 4(x^2-6x+9+y^2)$$

Expanding and simplifying yields

$$9x^2-24x+16 = 4x^2-24x+36+4y^2$$

$$5x^2-4y^2 = 20$$

5 Let (x, y) be the coordinates of point P. If $FP = 2MP$

$$\sqrt{(x-5)^2+y^2} = 2\sqrt{(x+1)^2}$$

Squaring both sides

$$(x-5)^2+y^2 = 4(x+1)^2$$

$$x^2-10x+25+y^2 = 4(x^2+2x+1)$$

$$x^2-10x+25+y^2 = 4x^2+8x+4$$

$$0 = 3x^2+18x-y^2-21$$

Completing the square gives,

$$0 = 3(x^2+6x)-y^2-21$$

$$0 = 3(x^2+6x+9-9)-y^2-21$$

$$0 = 3((x+3)^2-9)-y^2-21$$

$$0 = 3(x+3)^2-y^2-48$$

$$\frac{(x+3)^2}{16} - \frac{y^2}{48} = 1.$$

This is a hyperbola with centre $(-3, 0)$

6 Let (x, y) be the coordinates of point P. If $FP = 2MP$

$$\sqrt{x^2 + (y+1)^2} = 2\sqrt{(y+4)^2}$$

Squaring both sides

$$x^2 + (y+1)^2 = 4(y+4)^2$$
$$x^2 + y^2 + 2y + 1 = 4(y^2 + 8y + 16)$$
$$x^2 + y^2 + 2y + 1 = 4y^2 + 32y + 64$$
$$0 = 3y^2 + 30y - x^2 + 63$$

Completing the square gives,

$$0 = 3(y^2 + 10y) - x^2 + 63$$
$$0 = 3(y^2 + 10y + 25 - 25) - x^2 + 63$$
$$0 = 3((y+5)^2 - 25) - x^2 + 63$$
$$0 = 3(y+5)^2 - 75 - x^2 + 63$$
$$0 = 3(y+5)^2 - x^2 - 12$$
$$\frac{(y+5)^2}{4} - \frac{x^2}{12} = 1.$$

This is a hyperbola with centre $(0, -5)$